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REAL STRUCTURES ON TORUS BUNDLES AND THEIR DEFORMATIONS

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ABSTRACT. We describe the family of real structures σ on principal holomorphic torus bundles X over tori, and prove its connectedness when the complex dimension is at most three. Hence follows that the differentiable type (more precisely, the orbifold fundamental group) determines the deformation type of the pair (X, σ) provided we have complex dimension at most three, fibre dimension one, and a certain 'reality' condition on the fundamental group is satisfied.

1. INTRODUCTION

Given an oriented compact differentiable manifold M, a quite general problem is to consider the space $\mathcal{C}(M)$ of compatible complex structures J on M. Its connected components are called "Deformation classes in the large (of complex structures on M)".

A complete answer is known classically for curves, and, by the work of Kodaira, extending Enriques' classification, for special algebraic surfaces.

Since $\mathcal{C}(M)$ is infinite dimensional, one can equivalently consider instead the finite dimensional complex analytic space $\mathfrak{T}(M)$, called Teichmüller space, corresponding to the quotient of $\mathcal{C}(M)$ by the group $Diff^0(M)$ of diffeomorphisms isotopic to the identity. Its local structure is described by the Kuranishi theory ([Ku65]).

Inside $\mathfrak{T}(M)$ one has the open set $\mathfrak{K}T(M)$ of complex structures admitting a Kähler metric.

Hironaka showed ([Hir62]) that $\mathfrak{K}T(M)$ fails to be a union of connected components of $\mathfrak{T}(M)$.

Later, Sommese showed ([Somm75]) that, even in the case of complex tori of dimension ≥ 3, $\mathfrak{T}(M)$ has more than one connected component. It was known to Kodaira and Spencer that in this case $\mathfrak{K}T(M)$ is connected and coincides with the subset of translation invariant complex structures.

It seems to be more the rule than the exception that $\mathfrak{T}(M)$ has a lot of connected components. For instance, already for surfaces of general type it was shown by Manetti ([Man01]), and by Catanese and Wajnryb in the simply

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connected case ([C-W04]), that the number of connected components of $\mathfrak{T}(M)$ may be arbitrarily large.

The above work shows also that perhaps a more approachable version of the general problem is the following:

QUESTION : given a complex structure $X := (M, J)$, determine the connected component $\mathcal{C}^0(X)$ of $\mathcal{C}(M)$ containing it. I.e., given a complex manifold X, find its deformations in the large.

In [Cat02] (cf. also [Cat04]) it was shown that every deformation in the large of a complex torus is a complex torus. This was used to determine the stability by deformations in the large of holomorphic torus bundles over curves of genus at least two ([Cat04]), and to obtain the same result ([C-F05]) for threefolds which are holomorphic torus bundles over two dimensional tori, provided that the fundamental group satisfies a suitable 'reality' condition.

Consider now, as customary, a real manifold as a pair (X, σ), where X is a compact complex manifold and σ is a antiholomorphic involution.

Definition 1.1. *(X, σ) is said to be* **simple** *if, once we fix the differentiable type of the pair (X, σ), then we have a unique deformation class.*

I.e., M being the oriented differentiable manifold underlying X, simplicity holds iff the space $\mathcal{R}(M, \sigma)$ of compatible complex structures (which make σ antiholomorphic) is connected.

(X, σ) is said to be **quasi- simple** *if the above holds when we restrict ourselves to complex structures which are deformations of X (cf.* [DK02]*). I.e., (X, σ) is quasi- simple iff $\mathcal{R}(M, \sigma) \cap \mathcal{C}^0(X)$ is connected.*

More generally, as pointed out to us by Itenberg (cf. [DIK04]*), one can pose the problem of quasi-simplicity for the more general case of a finite subgroup G of the Klein group of dianalytic (i.e., biholomorphic or antibiholomorphic) automorphisms of X ($G \cong \mathbb{Z}/2$ in the previous case).*

A basic question here is : which real manifolds are simple ? Which ones are quasi simple ?

The answer is positive for curves, by work of Seppäla and Silhol ([S-S89]), and also for many surfaces of special type, thanks to work of many authors, like Comessatti, Silhol, Nikulin, Degtyarev, Itenberg, Kharlamov, Catanese-Frediani, Welschinger. These works (cf. references) give evidence to the following

CONJECTURE 3.2 Special surfaces are simple.

Motivated by this conjecture, in [C-F03] we introduced the notion of orbifold fundamental group in order to prove the simplicity of real hyperelliptic surfaces and in order to achieve a complete classification of them (there are 78 types).

The orbifold fundamental group of a real variety (X, σ) is an exact sequence

$$1 \to \pi_1(X) \to \Pi_\sigma := \pi_1^{orb}(X) \to \mathbb{Z}/2 \to 1,$$

where, if the action of σ is fixpoint free, Π_σ is the fundamental group of the quotient X/σ.

We are mainly interested here in the special case where X is a $K(\pi, 1)$ and more precisely where the universal cover $\tilde{X}$ of X is contractible. In this case we have

$$X = \tilde{X}/\pi_1(X), \;\; X/\sigma = \tilde{X}/\Pi_\sigma \; .$$

In the even more special case where $\tilde{X}$ has no moduli, the simplicity or quasi simplicity question is translated into the question whether the conjugacy class of the embedding $\Pi_\sigma \to Dian(\tilde{X})$ is parametrized by a connected variety.

Now, the cases which we treated already, real hyperelliptic surfaces and Kodaira surfaces ([C-F03] , [F04]) are both torus bundles over tori, whose study is reduced ([Cat04]) to the study of P.H.T.B.T. = principal holomorphic torus bundles over tori. The basic question which is of interest to us is to determine when a real P.H.T.B.T. is simple.

One could consider in greater generality the following situation: we have a nilpotent group Π, such that the factors of the central series are torsion free abelian of even rank.

Then the real Lie group $\Pi \otimes \mathbb{R}$ contains Π as a cocompact lattice (i.e., as a discrete subgroup with compact quotient $M := (\Pi \otimes \mathbb{R})/\Pi$) and we consider the differentiable manifold $M := (\Pi \otimes \mathbb{R})/\Pi$.

Already the case of tori has warned us that there could be too many exotic complex structures, so we restrict ourselves to consider only the right invariant complex structures on the homogeneous space M.

Obviously, the right invariant almost complex structures J correspond exactly to the complex structures on the tangent space at the origin, but there remains to write down explicitly the integrability conditions for a given almost complex structure J.

This is rather simple in the case of P.H.T.B.T. , treated in [Cat04] and in [C-F05]: we have an analogue of the Riemann bilinear relations (classically known for line bundles on complex tori) which are easy to explain as follows (cf. the next section for more details).

Consider the central extension of fundamental groups given by the homotopy exact sequence of the fibre bundle $f : X \to Y$:

$$1 \to \pi_1(T) := \Lambda \to \Pi := \pi_1(X) \to \pi_1(Y) := \Gamma \to 1.$$

The central extension is classified by an element

$$A \in H^2(\Gamma, \Lambda) \cong \Lambda^2(\Gamma^\vee) \otimes \Lambda.$$

As differentiable manifolds the base $Y = (\Gamma \otimes \mathbb{R})/\Gamma$, and the fibre $T = (\Lambda \otimes \mathbb{R})/\Lambda$, acquire a varying complex structure by writing a Hodge decomposition

$$\Gamma \otimes \mathbb{C} = V \oplus \bar{V}, \quad \Lambda \otimes \mathbb{C} = U \oplus \bar{U}.$$

The Riemann bilinear relations (which are equivalent to the integrability of the given almost complex structure on $M := (\Pi \otimes \mathbb{R})/\Pi$) amount to saying that the component of A in $\Lambda^2(V^\vee) \otimes \bar{U}$ is zero. These equations characterize then the so-called Appell Humbert family of such bundles, which parametrizes all the right invariant complex structures on $M := (\Pi \otimes \mathbb{R})/\Pi$.

The Riemann relations allow a decomposition

$$A = B + \overline{B}, \quad B = B' + B'', \; B' \in \Lambda^2(V)^\vee \otimes U, \; B'' \in (V^\vee \otimes \overline{V})^\vee \otimes U.$$

B' is called the holomorphic part, B'' is called the Hermitian part, and this decomposition reduces to questions of multilinear algebra the description of the spaces of holomorphic forms on X, of the subspace of closed holomorphic forms. and the explanation of an important phenomenon discovered by Nakamura ([Nak75]), namely the fact that local deformations of a parallelizable manifold need not be parallelizable.

One has for instance the following

Theorem, [Cat04]**:** X is parallelizable if and only if $B'' = 0$.

The Riemann bilinear relations define a parameter space which parametrizes the complex structures in the Appell Humbert family, and which is denoted by $\mathcal{T}'\mathcal{B}_A$, since it parametrizes torus bundles.

The parameter space $\mathcal{T}'\mathcal{B}_A$ is smooth if $m = dim_{\mathbb{C}}(Y) \leq 2, d = dim_{\mathbb{C}}(T) = 1$ (and it is singular already for $d = 1, m \geq 3$, at the points where $B'' = 0$).

The smoothness of $\mathcal{T}'\mathcal{B}_A$ makes it possible to analyse the versality of the (complete) Appell-Humbert family and to prove it ([C-F05]) under the following

'Reality' condition for A in the case $d = 1$: choose a basis e_1, e_2 of Λ and write $A = A_1 \otimes e_1 + A_2 \otimes e_2$. Then we want a real solution for the equation

$$Pfaffian(\lambda_1 A_1 + \lambda_2 A_2) = 0.$$

Theorem B , [C-F05] *The Appell Humbert family yields an open subset of $\mathcal{C}((\Pi \otimes \mathbb{R})/\Pi)$ if $d = 1, m = 2$ and moreover*

i) A is non degenerate

ii) $dim(ImA) = 2$

iii) A satisfies the 'reality' condition.

Combined with

Theorem A , [Cat04] *The Appell Humbert family yields a closed subset of $\mathcal{C}((\Pi \otimes \mathbb{R})/\Pi)$ if $d = 1$ and moreover $dim(ImA) = 2$.*

we find that, under the hypotheses of Theorem B, the Appell Humbert family yields a connected component of $\mathcal{C}((\Pi \otimes \mathbb{R})/\Pi)$.

In this paper we describe completely, via explicit equations, the family which parametrizes real structures on a torus bundle in the Appell Humbert family and then we show the connectivity of the family in special cases.

We obtain for instance the following result, which we formulate here in the less technical form

Theorem C *Assume again $d = 1, m = 2$, that A is nondegenerate, and fix the orbifold fundamental group Π_σ. Then the real structures for torus bundles in the Appell Humbert family are parametrized by a connected family.*

We obtain therefore as a corollary

Theorem D *Same assumptions as in theorem B: $d = 1, m = 2$, and i), ii), iii) are satisfied.*

Then simplicity holds for real torus bundles in the Appell Humbert family.

The paper is organized as follows : in section 2 we recall the theory developed in [Cat04] and [C-F05] concerning principal holomorphic torus bundles over tori, while section 3 is devoted to recalling the results already established for the Appell Humbert family.

The bulk of the paper is section 4, which studies the real structures on torus bundles in the Appell Humbert family. Strangely enough, the extra symmetries coming from the existence of a real structure makes calculations somewhat easier.

In fact, the real structure allows splittings

$$\Gamma \otimes \mathbb{R} = V^+ \oplus V^- \ , \ \Lambda \otimes \mathbb{R} = U^+ \oplus U^-,$$

and, after we show that the orbifold fundamental group Π_σ has a dianalytic affine representation in the complex vector space $(V \otimes \mathbb{C}) \oplus (U \otimes \mathbb{C})$, we can describe the complex structures in the Appell Humbert family which are compatible with the real structure as a pair of linear maps

$$B_2 : V^- \to V^+ \ , \ B_1 : U^- \to U^+ \ ,$$

satisfying explicit second degree matrix equations.

To illustrate the power of this way of calculating, we give a short new proof of the second author's result that simplicity holds for Kodaira surfaces.

Then we proceed to the case of threefolds, obtaining our main results (Theorems C and D).

2. Principal holomorphic torus bundles over tori: generalities

Throughout the paper, our set up will be the following: we have a holomorphic submersion between compact complex manifolds

$$f : X \to Y,$$

such that the base Y is a complex torus, and one fibre F (whence all the fibres, by theorem 2.1 of [Cat04]) is also a complex torus.

We shall denote this general situation by saying that f is differentiably a torus bundle.

We let $n = dimX$, $m = dimY$, $d = dimF = n - m$.

In general (cf.[FG65])) f is a holomorphic bundle if and only if all the fibres are biholomorphic.

This holds necessarily in the special case $d = 1$, because the moduli space for 1-dimensional complex tori exists and is isomorphic to $\mathbb{C}$.

Assume now more specifically that we have a holomorphic torus fibre bundle, thus we have (cf. [Cat04], pages 271-273) the exact sequence of holomorphic vector bundles

$$0 \to \Omega^1_Y \to f_*\Omega^1_X \to f_*\Omega^1_{X|Y} \to 0.$$

We have a principal holomorphic bundle if moreover $f_*\Omega^1_{X|Y}$ is a trivial holomorphic bundle.

Remark 2.1. *In general (cf. e.g.* [BPV84]*) if T is a complex torus, we have an exact sequence of complex Lie groups*

$$0 \to T \to Aut(T) \to M \to 1$$

where M is discrete. Taking sheaves of germs of holomorphic maps with source Y we get

$$0 \to \mathcal{H}(T)_Y \to \mathcal{H}(Aut(T))_Y \to M \to 1$$

hence the exact sequence

$$0 \to H^1(Y, \mathcal{H}(T)_Y) \to H^1(Y, \mathcal{H}(Aut(T)))_Y \to H^1(Y, M) \to H^2(Y, \mathcal{H}(T)_Y).$$

Since holomorphic bundles with base Y and fibre T are classified by the cohomology group $H^1(Y, \mathcal{H}(Aut(T))_Y)$, $H^1(Y, M)$ determines the discrete obstruction for a holomorphic bundle to be a principal holomorphic bundle.

In view of this, the study of holomorphic torus bundles is reduced to the study of principal holomorphic torus bundles.

In the case of a principal holomorphic bundle we write $\Lambda := \pi_1(T)$, $\Gamma := \pi_1(Y)$ and the exact sequence

$$\to H^0(\mathcal{H}(T)_Y) \to H^1(Y, \Lambda) \to H^1(Y, \mathcal{O}_Y^d) \to H^1(\mathcal{H}(T)_Y) \to^c \to H^2(Y, \Lambda)$$

determines a cohomology class $\epsilon \in H^2(Y, \Lambda)$ which classifies the central extension

$$1 \to \pi_1(T) = \Lambda \to \Pi := \pi_1(X) \to \pi_1(Y) = \Gamma \to 1$$

(it is central by the triviality of the monodromy automorphism).

Proposition 2.2. [C-F05] *Let $f : X \to Y$ be a principal holomorphic torus bundle over a torus as above.*

Then the universal covering of X is isomorphic to $\mathbb{C}^{m+d}$ and X is biholomorphic to a quotient $X \cong \mathbb{C}^{m+d}/\Pi$.

Let us briefly recall the classical way to look at the family $\mathcal{T}_m$ of complex tori of complex dimension $= m$. We fix a lattice Γ of rank $2m$, and we look at the complex (m-dimensional) subspaces $V \subset \Gamma \otimes \mathbb{C}$ such that $V \oplus \bar{V} = \Gamma \otimes \mathbb{C}$: to V corresponds the complex torus $Y_V := (\Gamma \otimes \mathbb{C})/(\Gamma \oplus \bar{V})$. We finally select one of the two resulting connected components by requiring that the complex orientation of V induces on $\Gamma \cong p_V(\Gamma)$ a fixed orientation.

We consider similarly the complex tori $T_U := (\Lambda \otimes \mathbb{C})/(\Lambda \oplus \bar{U})$ which can occur as fibres of f.

Consider now our principal holomorphic torus bundle $f : X \to Y$ over a complex torus Y_V of dimension m, and with fibre a complex torus T_U of dimension d and let $\epsilon \in H^2(Y, \Lambda) = H^2(\Gamma, \Lambda)$ be the cohomology class classifying the central extension

$$1 \to \Lambda \to \Pi \to \Gamma \to 1. \tag{1}$$

We reproduce an important result from [C-F05]

Lemma 2.3. *It is possible to "tensor" the above exact sequence with $\mathbb{R}$, obtaining an exact sequence of Lie Groups*

$$1 \to \Lambda \otimes \mathbb{R} \to \Pi \otimes \mathbb{R} \to \Gamma \otimes \mathbb{R} \to 1$$

such that Π is a discrete subgroup of $\Pi \otimes \mathbb{R}$ and such that X is diffeomorphic to the quotient manifold

$$M := \Pi \otimes \mathbb{R}/\Pi.$$

Proof. Consider, as usual, the map

$$A : \Gamma \times \Gamma \to \Lambda,$$

$$A(\gamma, \gamma') = [\hat{\gamma}, \hat{\gamma'}] = \hat{\gamma}\hat{\gamma'}(\hat{\gamma})^{-1}(\hat{\gamma'})^{-1},$$

where $\hat{\gamma}$ and $\hat{\gamma'}$ are respective liftings to Π of elements $\gamma, \gamma' \in \Gamma$. We observe that since the extension (1) is central, the definition of A does not depend on the choice of the liftings of γ, resp. γ' to Π.

As it is well known, A is bilinear and alternating, so A yields a cocycle in $H^2(\Gamma, \Lambda)$ which "classifies" the central extension (1). Let us review how does this more precisely hold.

Assume that $\{\gamma_1, ..., \gamma_{2m}\}$ is a basis of Γ and choose fixed liftings $\hat{\gamma}_i$ of γ_i in Π, for each $i = 1, \dots, 2m$. Then automatically we have determined a canonical way to lift elements $\gamma \in \Gamma$ through:

$$\gamma = \gamma_1^{n_1} \dots \gamma_{2m}^{n_{2m}} \mapsto \hat{\gamma} := \hat{\gamma}_1^{n_1} ... \hat{\gamma}_{2m}^{n_{2m}}.$$

Hence a canonical way to write the elements of Π as products $\lambda\hat{\gamma}$, where $\lambda \in \Lambda$ and $\hat{\gamma}$ is as above.

Since $\forall i, j$, one has

$$\hat{\gamma}_i\hat{\gamma}_j = A(\gamma_i, \gamma_j)\hat{\gamma}_j\hat{\gamma}_i,$$

we have a standard way of computing the products $(\lambda\hat{\gamma})(\lambda'\hat{\gamma}')$ as $\lambda''(\widehat{(\gamma\gamma')})$, where λ'' will be computed using A.

We can also view Π as a group of affine transformations of $(\Lambda \otimes \mathbb{R}) \oplus (\Gamma \otimes \mathbb{R})$. In fact, $(\Lambda \otimes \mathbb{R}) \oplus (\Gamma \otimes \mathbb{R})$ is a real vector space with basis $\{\lambda_1, ..., \lambda_{2d}, \gamma_1, ..., \gamma_{2m}\}$ where $\{\lambda_1, ..., \lambda_{2d}\}$ is a basis of Λ and the action of Π on $(\Lambda \otimes \mathbb{R}) \oplus (\Gamma \otimes \mathbb{R})$ is given as follows:

λ_i acts on $(\Lambda \otimes \mathbb{R}) \oplus (\Gamma \otimes \mathbb{R})$ sending (y, x) to $(y + \lambda_i, x)$, while the action of $\hat{\gamma}_j$ is defined using the multiplication $(\lambda\hat{\gamma}) \mapsto (\lambda\hat{\gamma})\hat{\gamma}_j$.

More precisely if $y \in \Lambda \otimes \mathbb{R}$, $x = \sum x_j\gamma_j \in \Gamma \otimes \mathbb{R}$, $\gamma' = \sum \nu_h\gamma_h \in \Gamma$, we have

$$(y, x)\hat{\gamma'} := (y + \phi_{\gamma'}(x), x + \gamma'),$$

where

$$\phi_{\gamma'}(x) = \sum_{j \geq h} x_j\nu_h A(\gamma_j, \gamma_h) = \sum_{j \geq h} x_j A_{jh}\nu_h =^t xT^-\gamma',$$

where T^- is the lower triangular part of the matrix A, so that we can write $A = T^- -^t T^-$.

Therefore we can endow $(\Lambda\otimes\mathbb{R})\oplus(\Gamma\otimes\mathbb{R}) =: \Pi\otimes\mathbb{R}$ with a Lie group structure defined by

$$(y,x)(y',x') = (y+y'+T^-(x,x'), x+x'),$$

and the quotient $(\Pi\otimes\mathbb{R})/\Pi$ of this Lie group by the discrete subgroup Π is immediately seen to be diffeomorphic to X.

Q.E.D.

Remark 2.4. *We can change coordinates in* $(\Lambda\otimes\mathbb{R})\oplus(\Gamma\otimes\mathbb{R})$ *in such a way that the action of the set* $\hat\Gamma\cong\Gamma\subset\Gamma\otimes\mathbb{R}$ *on* $\Pi\otimes\mathbb{R}$ *is given by*

$$(y,x)\hat\gamma = (y + A(x,\gamma) + 2S(\gamma,\gamma), x+\gamma),$$

where $S(\gamma,\gamma')$ *is a symmetric bilinear* $(\frac{1}{4}\Lambda)$*- valued form, and* $2S(\gamma,\gamma)\in\Lambda$*.*

Proof.

Let us define the symmetric form $S := -\frac{T^- + {}^tT^-}{4}$, so that $T^- + 2S = \frac{T^- - {}^tT^-}{2} = \frac{A}{2}$.

Consider the map $\Psi : (\Lambda\otimes\mathbb{R})\oplus(\Gamma\otimes\mathbb{R}) \to (\Lambda\otimes\mathbb{R})\oplus(\Gamma\otimes\mathbb{R})$ defined by $\Psi(y,x) = (2(y+S(x,x)), x) =: (\eta, x)$.

Then $\forall\hat\gamma\in\hat\Gamma$ we have an induced action

$$(\eta,x)\hat\gamma = \Psi((y,x)\hat\gamma) = \Psi(y+T^-(x,\gamma), x+\gamma) =$$

$$(2y + 2T^-(x,\gamma) + 2S(x+\gamma,x+\gamma), x+\gamma) = (\eta + A(x,\gamma) + 2S(\gamma,\gamma), x+\gamma),$$

and we conclude observing that $2S(\gamma,\gamma) = T^-(\gamma,\gamma)\in\Lambda$.

Q.E.D.

We recall from [Cat04] the First Riemann bilinear Relation: it is derived from the exact cohomology sequence

$$H^1(Y,\mathcal{O}_Y\otimes U)\cong H^1(Y,\mathcal{H}(U)_Y)\to H^1(\mathcal{H}(T)_Y)\to^c\to H^2(Y,\Lambda)\to H^2(Y,\mathcal{H}(U)_Y)$$

and says that the class ϵ maps to zero in $H^2(Y,\mathcal{H}(U)_Y)$. More concretely the

First Riemann Relation for principal holomorphic Torus Bundles

is expressed as follows:

Let $A:\Gamma\times\Gamma\to\Lambda$ **be the alternating bilinear map representing the cohomology class** ϵ **: then**

$$A\in\Lambda^2(\Gamma\otimes\mathbb{R})^\vee\otimes(\Lambda\otimes\mathbb{R})\subset\Lambda^2(\Gamma\otimes\mathbb{C})^\vee\otimes(\Lambda\otimes\mathbb{C})\subset\Lambda^2(V\oplus\bar V)^\vee\otimes(U\oplus\bar U),$$

satisfies the property that its component in $\Lambda^2(\bar V)^\vee\otimes(U)$ **is zero.**

It is important, for the forthcoming calculations, to understand in detail the bilinear algebra underlying the Riemann bilinear relation.

We observe preliminarly that one has a natural isomorphism $\Lambda^2(V\oplus\overline{V})^\vee\cong\Lambda^2(V)^\vee\oplus(V^\vee\otimes\overline{V})^\vee\oplus\Lambda^2(\overline{V})^\vee$, where the middle summand embeds by the wedge product : $w'\otimes\bar w\mapsto 2w'\wedge\bar w = w'\otimes\bar w - \bar w\otimes w'$.

Consider the alternating bilinear form

$$A\in\Lambda^2(\Gamma\otimes\mathbb{C})^\vee\otimes(\Lambda\otimes\mathbb{C}) = \Lambda^2(V\oplus\overline{V})^\vee\otimes(U\oplus\overline{U}),$$

satisfying the first bilinear relation and let us write

$$A = B + \overline{B},$$

where $B \in \Lambda^2(\Gamma \otimes \mathbb{C})^\vee \otimes U$, and $\overline{B} \in \Lambda^2(\Gamma \otimes \mathbb{C})^\vee \otimes \overline{U}$.

By the first bilinear relation $B = B' + B''$, with $B' \in \Lambda^2(V)^\vee \otimes U$, $B'' \in (V^\vee \otimes \overline{V})^\vee \otimes U$.

Concretely, $A = B' + B'' + \overline{B'} + \overline{B''}$, where B' is an alternating complex bilinear form. The fact that B'' is alternating reads out as:

$$B''(v', \bar{v}) = -B''(\bar{v}, v') \; \forall v, v' \in V$$

whereas conjugation of tensors reads out as:

$$\bar{B}(\bar{x}, \bar{y}) = \overline{B(x,y)} \; \forall x, y \; \Rightarrow \bar{B}''(\bar{v}, v') = \overline{B''(v, \bar{v'})}.$$

3. Appell Humbert families

We shall recall in this section the definition of the family given by the pairs of subspaces satisfying the Riemann bilinear relations and some related results obtained in [Cat04], [C-F05].

Definition 3.1. *Given A as above, we define $\mathcal{TB}_A$ as the subset of the product of Grassmann Manifolds $Gr(m, 2m) \times Gr(d, 2d)$ defined by*

$$\mathcal{TB}_A = \{(V, U) \in Gr(m, 2m) \times Gr(d, 2d) \mid V \cap \overline{V} = (0),$$
$$U \cap \overline{U} = (0), \mid \text{the component of } A \textit{ in } \Lambda^2(\overline{V})^\vee \otimes U \textit{ is } = 0\}.$$

This complex space is called the Appell Humbert space of Torus Bundles.

Since this space is not connected, we restrict ourselves to its intersection with $\mathcal{T}_m \times \mathcal{T}_d$, i.e., we take two fixed orientations of Λ, resp. Γ, and consider pairs of complex structures which have the same orientation as the fixed ones.

One sees immediately that $\mathcal{TB}_A$ is a complex analytic variety of codimension at most $dm(m-1)/2$.

Note however that, for $d \geq 3, m >> 0$ we get a negative expected dimension. The structure of these complex spaces should be investigated in general, for our present purposes we recall from [C-F05] the following

Lemma 3.2. *(*[C-F05]*) If $d = 1$ then the open set $\mathcal{TB}_A \cap (\mathcal{T}_m \times \mathcal{T}_d)$ is connected.*

Definition 3.3. *The standard (Appell-Humbert) family of torus bundles parametrized by $\mathcal{TB}_A$ is the family of principal holomorphic torus bundles $X_{V,U}$ over $Y := Y_V$ and with fibre $T := T_U$ determined by the cocycle in $H^1(Y, \mathcal{H}(T)_Y)$ obtained by taking $f_\gamma(v)$ which is the class modulo Λ of*

$$F_\gamma(v) := B'(v, p_V(\gamma)) + 2B''(v, \overline{p_V(\gamma)}) + B''(p_V(\gamma), \overline{p_V(\gamma)}), \forall v \in V.$$

In other words, $X_{V,U}$ is the quotient of $T_U \times V$ by the action of Γ such that

$$\gamma([u], v) = ([u + B'(v, p_V(\gamma)) + 2B''(v, \overline{p_V(\gamma)}) + B''(p_V(\gamma), \overline{p_V(\gamma)})], v + p_V(\gamma)).$$

Remark 3.4. *As observed in* [C-F05] *the above formula was a correction of the formula given in Definition 6.4 of* [Cat04], *where an identification of* $\Gamma \otimes \mathbb{R}$ *with* V *was used, and thus* $A(z,\gamma)$ *was identified with* $B(z,\gamma)$. *In the latter formula one had thus*

$-B'(v,\gamma) - B''(v, \overline{p_V(\gamma)})$ *instead of*

$B'(v,\gamma) + 2B''(v, \overline{p_V(\gamma)}) + B''(p_V(\gamma), \overline{p_V(\gamma)})$.

We also recall from [Cat04] the definition of the complete Appell-Humbert space.

Definition 3.5. *Given* A *as above we define*

$$\mathcal{T}'\mathcal{B}_A = \{(V,U,\phi) \mid (V,U) \in \mathcal{TB}_A, \ \phi \in H^1(Y_V, \mathcal{H}(U)_{Y_V}) \cong \overline{V}^\vee \otimes U\}.$$

The complete Appell-Humbert family of torus bundles parametrized by $\mathcal{T}'\mathcal{B}_A$ *is the family of principal holomorphic torus bundles* $X_{V,U,\phi}$ *on* $Y := Y_V$ *and with fibre* $T := T_U$ *determined by the cocycle in* $H^1(Y, \mathcal{H}(T)_Y)$ *obtained by taking the sum of* $f_\gamma(z)$ *with the cocycle* $\phi \in H^1(Y_V, \mathcal{H}(U)_{Y_V}) \cong H^1(Y, \mathcal{O}_Y^d)$.

Finally we have the following

Theorem 3.6. [Cat04]

Any principal holomorphic torus bundle with extension class isomorphic to $\epsilon \in H^2(\Gamma, \Lambda)$ *occurs in the complete Appell-Humbert family* $\mathcal{T}'\mathcal{B}_A$.

We recall from [C-F05] also the following

Proposition 3.7. [C-F05] *Let* $A : \Gamma \times \Gamma \to \Lambda$ *be non zero.*

If $m = 2, d = 1$, *i.e.,* $\Gamma \cong \mathbb{Z}^4$, $\Lambda \cong \mathbb{Z}^2$, *both Appell - Humbert spaces* $\mathcal{TB}_A$ *and* $\mathcal{T}'\mathcal{B}_A$ *are smooth.*

If $d = 1$ *and* $m \geq 3$, $\mathcal{TB}_A$ *is singular at the points where* $B'' = 0$.

4. Symmetries of principal holomorphic torus bundles over tori

Let us now assume that $f : X \to Y$ is a real principal holomorphic torus bundle over a torus, and let $\sigma : X \to X$ be an antiholomorphic involution. We assume as above that $Y = Y_V$, $V \cong \mathbb{C}^m$, while the fibre of f is $T = T_U$, with $U \cong \mathbb{C}^d$. In 2.2 we have seen that the universal covering of X is isomorphic to $V \oplus U \cong \mathbb{C}^{m+d}$ and we can find a lifting $\tilde{\sigma}$ of σ to the universal covering $U \oplus V \cong \mathbb{C}^{m+d}$.

Proposition 4.1. *Assume that the alternating form* $A : \Gamma \times \Gamma \to \Lambda$ *is nondegenerate, or equivalently that* $\Lambda = Z(\Pi)$. *Let* $\tilde{\sigma}$ *be a lifting of* σ *to* $U \oplus V \cong \mathbb{C}^{m+d}$: *then* $\tilde{\sigma}$ *is an affine transformation.*

Proof. Since $\tilde{\sigma}$ is a lifting of σ to the universal covering, $\tilde{\sigma}$ acts by conjugation on the fundamental group Π of X, therefore it also acts on the centre Λ of Π, because it is characteristic.

Hence for every $\lambda \in \Lambda$, there exists a $\lambda' \in \Lambda$ such that

$$\tilde{\sigma}(u + p_U(\lambda), v) = \tilde{\sigma}(u,v) + p_U(\lambda'),$$

where $p_U : U \oplus \bar{U} \to U$ denotes as usual the projection on the first factor. Assume that $\tilde{\sigma} : U \oplus V \to U \oplus V$ is given by $\tilde{\sigma}(u, v) = (\sigma_1(u, v), \sigma_2(u, v))$. Then we must have

$$(\sigma_1(u + p_U(\lambda), v), \sigma_2(u + p_U(\lambda), v)) = (\sigma_1(u, v) + p_U(\lambda'), \sigma_2(u, v)),$$

therefore σ_2 is constant as a function of u, and we may write $\sigma_2(u, v) = \sigma_2(v)$.

Looking at the first component we obtain $\sigma_1(u+p_U(\lambda), v) = \sigma_1(u, v)+p_U(\lambda')$, so σ_1 is affine antiholomorphic in u and we may write

$$\sigma_1(u, v) = A_1(v)\bar{u} + c_1(v) \tag{2}$$

where $A_1(v)$ is a linear map depending antiholomorphically on v (we may think of it as a $(d \times d)$ matrix once we fix a basis for U).

Recall that, for every $\gamma \in \Gamma$, and for any lift to $U \oplus V$ of the action of γ on $T \times V$, there exists $\lambda' \in \Lambda$ such that

$$\gamma(u, v) = (u + F_\gamma(v) + p_U(\lambda'), v + p_V(\gamma)),\ \forall (u, v) \in U \oplus V,$$

where $F_\gamma(v)$ is as in definition (3.3) and $p_V : V \oplus \bar{V} \to V$ is the first projection. For all $\gamma \in \Gamma$ there must therefore exist $\gamma' \in \Gamma$, and a $\lambda'' \in \Lambda$ such that $\tilde{\sigma} \circ \gamma = \lambda''\gamma' \circ \tilde{\sigma}$. Hence we have

$$\begin{aligned}\tilde{\sigma} \circ \gamma(u, v) &= \tilde{\sigma}(u + F_\gamma(v) + p_U(\lambda'), v + p_V(\gamma)) = \\ &= (\sigma_1(u + F_\gamma(v) + p_U(\lambda'), v + p_V(\gamma)), \sigma_2(v + p_V(\gamma))) = \\ &= \lambda''\gamma'(\tilde{\sigma}(u, v)) = (\sigma_1(u, v) + F_{\gamma'}(\sigma_2(v)) + p_U(\lambda'''), \sigma_2(v) + p_V(\gamma')),\end{aligned}$$

where $\lambda''' \in \Lambda$. Therefore we obtain that σ_2 is affine antiholomorphic,

$$\sigma_2(v) = A_2\bar{v} + d_2,$$

where A_2 is a linear map (a $(m \times m)$ matrix if we fix a basis of V) and we have

$$A_2\bar{v} + A_2\overline{p_V(\gamma)} = A_2\bar{v} + p_V(\gamma'),$$

so that

$$A_2\overline{p_V(\gamma)} = p_V(\gamma').$$

Looking at the first component we have

$$\sigma_1(u + F_\gamma(v) + p_U(\lambda'), v + p_V(\gamma)) = \sigma_1(u, v) + F_{\gamma'}(A_2\bar{v} + d_2) + p_U(\lambda''').$$

Now using (2) we have

$$A_1(v+p_V(\gamma))(\bar{u}+\overline{F_\gamma(v)}+\overline{p_U(\lambda')})+c_1(v+p_V(\gamma)) = A_1(v)\bar{u}+c_1(v)+F_{\gamma'}(A_2\bar{v})+F_{\gamma'}(d_2)+p_U(\lambda'''). \tag{3}$$

So by derivation with respect to the variables $\bar{u}_i's$ we obtain

$$A_1(v + p_V(\gamma)) = A_1(v),$$

for all $v \in V$, for all $\gamma \in \Gamma$, therefore A_1 is constant in v and we can write $A_1(v) = A_1$. Now (3) becomes

$$c_1(v+p_V(\gamma))-c_1(v) = F_{A_2\overline{p_V(\gamma)}}(A_2\bar{v})-A_1\overline{F_\gamma(v)}+F_{A_2\overline{p_V(\gamma)}}(d_2)+p_U(\lambda''')-A_1\overline{p_U(\lambda')} \tag{4}$$

and derivation with respect to the variables $\bar{v}_j's$ yields the vanishing of the derivatives of

$$c_1(v + p_V(\gamma)) - c_1(v),\ \forall v \in V,\ \forall \gamma \in \Gamma,$$

so these derivatives are constant and

$$c_1(v) = q(v,v) + l(v) + d_1,$$

where $q(v,v)$ is quadratic in v, $l(v)$ is linear in v, and d_1 is a constant.

Now (4) gives

$$q(v+p_V(\gamma), v+p_V(\gamma)) + l(v) + l(p_V(\gamma)) + d_1 - q(v,v) - l(v) - d_1 =$$
$$= F_{A_2\overline{p_V(\gamma)}}(A_2\bar{v}) - A_1\overline{F_\gamma(v)} + F_{A_2\overline{p_V(\gamma)}}(d_2) + p_U(\lambda''') - A_1\overline{p_U(\lambda')},$$

so

$$q(p_V(\gamma), p_V(\gamma)) + 2q(v, p_V(\gamma)) + l(p_V(\gamma)) =$$
$$= F_{A_2\overline{p_V(\gamma)}}(A_2\bar{v}) - A_1\overline{F_\gamma(v)} + F_{A_2\overline{p_V(\gamma)}}(d_2) + p_U(\lambda''') - A_1\overline{p_U(\lambda')}.$$

Now, by looking in the above expression at the variable γ we immediately get $q(p_V(\gamma), p_V(\gamma)) = 0$, since it is the only quadratic term (substitute γ with $m\gamma$ and look at the asymptotic growth). Since $q(p_V(\gamma), p_V(\gamma)) = 0$ for all $\gamma \in \Gamma$ and $p_V(\Gamma)$ is a lattice in V, we must have $q = 0$ and $c_1(v) = l(v)+d_1 = L\bar{v}+d_1$, where L is a $(d \times m)$ matrix. So we finally get

$$\tilde{\sigma}(u,v) = (A_1\bar{u} + L\bar{v} + d_1, A_2\bar{v} + d_2)$$

and the proposition is proven. Q.E.D.

Remark 4.2. *With the above notation we have*

$$\tilde{\sigma}(u,v) = \begin{pmatrix} A_1 & L \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} \bar{u} \\ \bar{v} \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

and the following properties hold:

(1) $A_1\overline{p_U(\Lambda)} = p_U(\Lambda)$, $A_2\overline{p_V(\Gamma)} = p_V(\Gamma)$.

(2) $\forall \gamma \in \Gamma$ *we have*

$$A_1 \overline{B''(p_V(\gamma), \overline{p_V(\gamma)})} = B''(A_2\overline{p_V(\gamma)}, \overline{A_2}p_V(\gamma)).$$

(3) $\forall v \in V$, $\forall \gamma \in \Gamma$ *we have*

$$A_1\overline{B'(v, p_V(\gamma))} + 2A_1 \overline{B''(v, \overline{p_V(\gamma)})} = B'(A_2\bar{v}, A_2\overline{p_V(\gamma)}) + 2B''(A_2\bar{v}, \overline{A_2}p_V(\gamma)).$$

(4) $\forall \gamma \in \Gamma$

$$L\overline{p_V(\gamma)} - B'(d_2, A_2\overline{p_V(\gamma)}) + 2B''(d_2, \overline{A_2}p_V(\gamma)) + B''(A_2\overline{p_V(\gamma)}, \overline{A_2}p_V(\gamma)) \in p_U(\Lambda).$$

(5) $A_1\bar{A}_1 = I$, $A_2\bar{A}_2 = I$.

(6) *There exists a* $\gamma \in \Gamma$ *such that* $\forall v \in V$

$$A_1\overline{L}(v) + L\overline{A_2}v = F_\gamma(v).$$

(7) $\sigma_2^2 = Id \ mod \ p_V(\Gamma)$, *i.e.* σ_2 *induces an antiholomorphic involution on* Y.

(8) $A_1\overline{d_1} + L\overline{d_2} + d_1 \in p_U(\Lambda)$.

Proof. Conditions (1), (2), (3) and (4) easily follow as in the proof of (4.1) by imposing that for all $\lambda \in \Lambda$ there exists a $\lambda' \in \Lambda$ such that $\tilde{\sigma} \circ \lambda = \lambda' \circ \tilde{\sigma}$, and for all $\gamma \in \Gamma$ there must exist $\gamma' \in \Gamma$, $\lambda'' \in \Lambda$ such that $\tilde{\sigma} \circ \gamma = \lambda''\gamma' \circ \tilde{\sigma}$. Here we also used the expression of $F_\gamma(v)$ given in definition (3.3):

$$F_\gamma(v) := B'(v, p_V(\gamma)) + 2B''(v, \overline{p_V(\gamma)}) + B''(p_V(\gamma), \overline{p_V(\gamma)}), \forall v \in V.$$

Conditions (5), (6), (7) and (8) immediately follow by imposing $\tilde{\sigma}^2 \in \Pi$. $\square$

Theorem 4.3. *Let $f : X \to Y$ be a principal holomorphic torus bundle over a torus such that A is non degenerate and assume that $\sigma : X \to X$ is an antiholomorphic involution on X. The differentiable type of the pair (X, σ) is completely determined by the orbifold fundamental group exact sequence. More precisely, the affine embedding of Π_σ is uniquely determined up to conjugation.*

Proof. Let

$$1 \to \Pi \to \hat{\Pi} := \Pi_\sigma \to \mathbb{Z}/2 \to 1 \tag{5}$$

be the orbifold fundamental group exact sequence of the pair (X, σ). Every lifting $\tilde{\sigma}$ of σ to $\hat{\Pi}$ acts by conjugation on Π, so it acts by conjugation on the centre Λ of Π and thus it acts on the quotient $\Gamma = \Pi/\Lambda$. Therefore we have determined an extension

$$1 \to \Gamma \to \hat{\Gamma} \to \mathbb{Z}/2 \to 1$$

which is the orbifold fundamental group exact sequence of the real torus (Y, σ_2), where σ_2 denotes as above the second component of $\tilde{\sigma}$. Since for a real torus the orbifold fundamental group exact sequence determines the differentiable type (cf. [Cat02]), we have shown that we can fix the differentiable type of the pair (Y, σ_2).

By proposition 4.1 we know that any lifting $\tilde{\sigma}$ of σ to the universal covering $(\Lambda \otimes \mathbb{R}) \oplus (\Gamma \otimes \mathbb{R})$ is of the form

$$\tilde{\sigma}(y, x) = \begin{pmatrix} A_1 & L \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

and we know the $(2m \times 2m)$-matrix A_2 and the translation vector d_2. We also have $A_1^2 = I$, $A_2^2 = I$, because $\tilde{\sigma}^2 \in \Pi$. Furthermore we know $\tilde{\sigma}^2 \in \Pi$, since we know the extension (5), therefore we know the vector

$$\begin{pmatrix} A_1 & L \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix} = \begin{pmatrix} Ld_2 + A_1 d_1 + d_1 \\ A_2 d_2 + d_2 \end{pmatrix}$$

which is the translation part of $\tilde{\sigma}^2$.

For every $\lambda \in \Lambda$ we know $\tilde{\sigma}\lambda\tilde{\sigma}^{-1} \in \Lambda$ and $\tilde{\sigma}\lambda\tilde{\sigma}^{-1}(w) = w + M\lambda$, $\forall w \in (\Gamma \otimes \mathbb{R}) \oplus (\Lambda \otimes \mathbb{R})$, where

$$M = \begin{pmatrix} A_1 & L \\ 0 & A_2 \end{pmatrix}$$

Thus we know $A_1(\lambda) \in \Lambda$, $\forall \lambda \in \Lambda$ and since Λ generates $\Lambda \otimes \mathbb{R}$, we know A_1. So we know A_1, A_2, d_2, and $Ld_2 + A_1 d_1 + d_1$.

For all $\hat{\gamma} \in \Pi$ lifting a given $\gamma \in \Gamma$, $\gamma \neq 0$, we know $\tilde{\sigma}\hat{\gamma}\tilde{\sigma}^{-1}$. Let us set $\tilde{\sigma}(w) = Mw + b$, $\hat{\gamma}(w) = Dw + h$, where

$$D \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} I_{2d} & \phi_\gamma \\ 0 & I_{2m} \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix}$$

where $\phi_\gamma(x)$ can be chosen to be equal to $A(x, \gamma)$ as it is proven in (2.4), and $h = \begin{pmatrix} l \\ \gamma \end{pmatrix}$, where $l \in \Lambda$.

We have $\tilde\sigma\hat\gamma\tilde\sigma^{-1}(w) = \tilde\sigma\hat\gamma(M^{-1}w - M^{-1}b) = \tilde\sigma(DM^{-1}w - DM^{-1}b + h) = MDM^{-1}w - MDM^{-1}b + Mh + b$. Thus we know MDM^{-1} and $-MDM^{-1}b + Mh + b$. One easily computes

$$-MDM^{-1}b + Mh + b = \begin{pmatrix} -A_1\phi_{\gamma'}A_2^{-1}d_2 + L\gamma + A_1 l \\ A_2\gamma \end{pmatrix}$$

and since we know both $-A_1\phi_{\gamma'}A_2^{-1}d_2$ and $A_1 l$, we also know $L\gamma$ for all $\gamma \in \Gamma$. Now Γ generates $\Gamma \otimes \mathbb{R}$, so we know $L : \Gamma \otimes \mathbb{R} \to \Lambda \otimes \mathbb{R}$.

We have already seen that we know $Ld_2 + A_1 d_1 + d_1$, so we also know $d_1' = A_1 d_1 + d_1$ and we know A_1. Now, $A_1^2 = I$, so we can decompose $\Lambda \otimes \mathbb{R} = Z \oplus S \oplus W^+ \oplus W^-$, where $W^\pm$ are the ± 1-eigenspaces of A_1 and where

$$A_1(z, s, w^+, w^-) = (s, z, w^+, -w^-).$$

Observe that since A_1 is antiholomorphic, W^+ and W^- have the same dimension (also $dim(Z) = dim(S)$). Thus we can write $d_1 = (z, s, d_1^+, d_1^-)$. $A_1(z, s, d_1^+, d_1^-) + (z, s, d_1^+, d_1^-) = (s + z, s + z, 2d_1^+, 0)$, so we know d_1^+ and $z + s$. We can change the origin by translating with $(y, x) \in (\Lambda \otimes \mathbb{R}) \oplus (\Gamma \otimes \mathbb{R}) \mapsto (y + t, x) \in (\Lambda \otimes \mathbb{R}) \oplus (\Gamma \otimes \mathbb{R})$ and we may assume that $d_1 = (z, s, d_1^+, d_1^-) + A_1(t) - t = (z, s, d_1^+, d_1^-) + (t_2 - t_1, t_1 - t_2, 0, -2t^-)$ ($t = (t_1, t_2, t^+, t^-)$). So we can choose $t^- = d_1^-/2$, $t_2 - t_1 = -z$, therefore the first and the last components of d_1 can be chosen equal to zero. Then the second component is $s + t_1 - t_2 = s + z$ and therefore we know it, finally the third component is d_1^+ and we already know it.

$\square$

Remark 4.4. *Let us fix the orbifold fundamental group exact sequence of a real principal holomorphic torus bundle over a torus such that the alternating bilinear form $A : \Gamma \times \Gamma \to \Lambda$ is non degenerate. Then the topological and differentiable action of Π_σ is fixed, and the action of any element $\tilde\sigma$ of the orbifold fundamental group $\hat\Pi$ on the universal covering $(\Lambda \otimes \mathbb{R}) \oplus (\Gamma \otimes \mathbb{R})$ is given by the affine transformation*

$$\tilde\sigma \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} A_1 & L \\ 0 & A_2 \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} + \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}$$

If we fix such an element $\tilde\sigma$, the following holds:

1. $A_1 : \Lambda \to \Lambda$, $A_1^2 = I$.
2. $A_2 : \Gamma \to \Gamma$, $A_2^2 = I$.
3. $\forall\gamma$, $\forall x \in \Gamma \otimes \mathbb{R}$, $A_1(A(x, \gamma)) = A(A_2(x), A_2(\gamma))$.
4. *The $\mathbb{R}$ - linear map $L' : \Gamma \otimes \mathbb{R} \to \Lambda \otimes \mathbb{R}$, $L'(x) := Lx - A(d_2, A_2(x))$ satisfies $L'(\Gamma) \subset \Lambda$.*
5. $A_2(d_2) + d_2 \in \Gamma$.
6. $L(d_2) + A_1(d_1) + d_1 \in \Lambda$.
7. $\exists\gamma \in \Gamma$ *such that* $L(A_2(x)) + A_1(L(x)) = -A(x, \gamma)$ $\forall x \in \Gamma \otimes \mathbb{R}$.

Proof. For all $\lambda \in \Lambda$ we have $\tilde\sigma\lambda\tilde\sigma^{-1} \in \Lambda$ and this immediately implies that $A_1(\Lambda) \subset \Lambda$. $A_1^2 = I$, since $\tilde\sigma^2 \in \Pi$ and for all $g \in \Pi$ the action of g on $(\Lambda \otimes \mathbb{R}) \oplus (\Gamma \otimes \mathbb{R})$ is given by

$$g \begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} I & \phi_\gamma \\ 0 & I \end{pmatrix} \begin{pmatrix} y \\ x \end{pmatrix} + \begin{pmatrix} l \\ \gamma \end{pmatrix}$$

where γ is the image of g in Γ, $l \in \Lambda$, and $\phi_\gamma(x) = A(x,\gamma)$ as in (2.4).

This also implies that $A_2^2 = Id$.

For every $g \in \Pi$, $g \notin \Lambda$, we know $\tilde\sigma g \tilde\sigma^{-1} = h \in \Pi - \Lambda$. If we set

$$g\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} I & \phi_\gamma \\ 0 & I \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} + \begin{pmatrix} l \\ \gamma \end{pmatrix}$$

where as above γ is the image of g in Γ and $l \in \Lambda$, and

$$h\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} I & \phi_\delta \\ 0 & I \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} + \begin{pmatrix} \lambda \\ \delta \end{pmatrix}$$

where δ is the image of h in Γ and $\lambda \in \Lambda$, we have

$$\begin{aligned}\tilde\sigma g \tilde\sigma^{-1}\begin{pmatrix} y \\ x \end{pmatrix} &= \begin{pmatrix} I & A_1\phi_\gamma A_2^{-1} \\ 0 & I \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} + \begin{pmatrix} -A_1\phi_\gamma A_2^{-1}d_2 + L\gamma + A_1 l \\ A_2\gamma \end{pmatrix} \\ &= h\begin{pmatrix} y \\ x \end{pmatrix} = \begin{pmatrix} I & \phi_\delta \\ 0 & I \end{pmatrix}\begin{pmatrix} y \\ x \end{pmatrix} + \begin{pmatrix} \lambda \\ \delta \end{pmatrix}\end{aligned}$$

This yields $A_1\phi_\gamma A_2^{-1} = \phi_\delta$, $A_2\gamma = \delta$, $-A_1\phi_\gamma A_2^{-1}d_2 + L\gamma + A_1 l = \lambda \in \Lambda$. So $A_2(\Gamma) = \Gamma$ and for all $\gamma \in \Gamma$ we have $L'\gamma = L\gamma - \phi_{A_2\gamma}(d_2) = L(\gamma) - A(d_2, A_2(\gamma)) \in \Lambda$.

The condition $A_1\phi_\gamma A_2^{-1} = \phi_{A_2(\gamma)}$, $\forall \gamma \in \Gamma$ can be written as

$$A_1(A(x,\gamma)) = A(A_2(x), A_2(\gamma)),\ \forall x \in \Gamma \otimes \mathbb{R},\ \forall \gamma \in \Gamma.$$

Finally $\tilde\sigma^2 \in \Pi$ implies that $A_2(d_2) + d_2 \in \Gamma$, $L(d_2) + A_1(d_1) + d_1 \in \Lambda$, furthermore there must exist a $\gamma \in \Gamma$ such that $LA_2 + A_1 L = \phi_\gamma$. This implies that, for every $\delta \in \Gamma$, $L(A_2(\delta)) + A_1(L(\delta)) = -A(\delta, -\gamma) \in \Lambda$. □

We have three $\mathbb{R}$ - linear maps $A_1 : \Lambda \to \Lambda$, $A_2 : \Gamma \to \Gamma$, $L : \Gamma \otimes \mathbb{R} \to \Lambda \otimes \mathbb{R}$ with the above properties.

Let us now fix a complex structure on the bundle $f : X \to Y$, i.e. we fix a point (V, U) in $\mathcal{TB}_A = \{(V,U) \in Gr(m, 2m) \times Gr(d, 2d) \mid V \cap \overline{V} = (0),\ U \cap \overline{U} = (0),\ \mid \textit{the component of } A \textit{ in } \Lambda^2(\overline{V})^\vee \otimes U \textit{ is } = 0\}$.

We may now see A_1 as a real element in $(\Lambda \otimes \mathbb{C})^\vee \otimes (\Lambda \otimes \mathbb{C}) = (U \oplus \overline{U})^\vee \otimes (U \oplus \overline{U})$ and we want to impose that A_1 is antiholomorphic with respect to the chosen complex structure, so we want that the component of A_1 in $U^\vee \otimes U$ is zero. Analogously we see A_2 as a real element in $(\Gamma \otimes \mathbb{C})^\vee \otimes (\Gamma \otimes \mathbb{C}) = (V \oplus \overline{V})^\vee \otimes (V \oplus \overline{V})$ and we want that the component of A_2 in $V^\vee \otimes V$ is zero. Finally we also see L as a real element in $(\Gamma \otimes \mathbb{C})^\vee \otimes (\Lambda \otimes \mathbb{C}) = (V \oplus \overline{V})^\vee \otimes (U \oplus \overline{U})$ and we want that its component in $V^\vee \otimes U$ is zero.

Observe that since any other lifting $\tilde\sigma'$ of σ to $\hat\Pi$ is obtained from $\tilde\sigma$ by composition with an element in Π, which acts holomorphically with respect to the chosen complex structure, we may give the following definition.

Definition 4.5. *We define spaces*

$$\mathcal{TB}_A^{\mathbb{R}}(A_1, A_2, L) = \{(V,U) \in Gr(m,2m) \times Gr(d,2d) \mid V \cap \overline{V} = (0),$$
$$U \cap \overline{U} = (0),\ \mid \textit{the component of } A \textit{ in } \Lambda^2(\overline{V})^\vee \otimes U \textit{ is } = 0,\ \textit{the component}$$
$$\textit{of } A_1 \textit{ in } U^\vee \otimes U \textit{ is } = 0,\ \textit{the component of } A_2 \textit{ in } V^\vee \otimes V \textit{ is } = 0,$$

$$\textit{the component of } L \textit{ in } V^\vee \otimes U \textit{ is } = 0\},$$

and

$$\mathcal{T}'\mathcal{B}^{\mathbb{R}}_A(A_1, A_2, L) = \{(V, U, \phi) \mid (V, U) \in \mathcal{T}\mathcal{B}^{\mathbb{R}}_A(A_1, A_2, L)\ \phi \in H^1(Y_V, \mathcal{H}(U)_{Y_V}) \cong \overline{V}^\vee \otimes U\}.$$

Remark 4.6. *Assume that we have fixed the orbifold fundamental group exact sequence of the pair (X, σ) and that the alternating bilinear form $A : \Gamma \times \Gamma \to \Lambda$ is non degenerate: then the above spaces parametrize families of real structures on principal holomorphic torus bundles, and in particular the latter family parametrizes all the possible real structures on a principal holomorphic torus bundle over a torus with a given real topological type.*

Let us now write

$$\Lambda \otimes \mathbb{R} = U^+ \oplus U^-,$$
$$\Gamma \otimes \mathbb{R} = V^+ \oplus V^-,$$

according to the eigenspace decomposition for A_1, respectively for A_2.

If we set $A = A^+ + A^-$, where A^+ is the component of A with values in U^+, A^- is the component of A with values in U^-, we can write condition (3) of (4.4) as follows: $\forall x \in \Gamma \otimes \mathbb{R}$, $\forall \gamma \in \Gamma$,

$$A_1(A(x, \gamma)) = A^+(x, \gamma) - A^-(x, \gamma) = A(A_2(x), A_2(\gamma)) = A(x^+ - x^-, \gamma^+ - \gamma^-),$$

where $x = x^+ + x^-$, $\gamma = \gamma^+ + \gamma^-$ according to the above decomposition.

Thus we have

$$A^+(x^+ + x^-, \gamma^+ + \gamma^-) = A^+(x^+ - x^-, \gamma^+ - \gamma^-),$$
$$A^-(x^+ + x^-, \gamma^+ + \gamma^-) = -A^-(x^+ - x^-, \gamma^+ - \gamma^-),$$

which in turn is equivalent to

$$A^+(x^+, \gamma^-) + A^+(x^-, \gamma^+) = 0, \quad A^-(x^+, \gamma^+) + A^-(x^-, \gamma^-) = 0.$$

We conclude then that condition (3) is equivalent to

$$A^+|_{V^+ \times V^-} \equiv 0,\ A^-|_{V^+ \times V^+} \equiv 0,\ A^-|_{V^- \times V^-} \equiv 0,\ \textit{i.e.}, \tag{6}$$

$$A(x_1, x_2) = A^+(x_1^+, x_2^+) + A^+(x_1^-, x_2^-) + A^-(x_1^-, x_2^+) + A^-(x_1^+, x_2^-). \tag{7}$$

Condition (7) of (4.4) is equivalent to the existence of $\hat{\gamma} \in \Gamma$ such that

$$L^+(x^+ - x^-) + L^-(x^+ - x^-) + L^+(x^+ + x^-) - L^-(x^+ + x^-) =$$
$$-A^+(x^+ + x^-, \hat{\gamma}^+ + \hat{\gamma}^-) - A^-(x^+ + x^-, \hat{\gamma}^+ + \hat{\gamma}^-),$$

that is, to

$$2L^+(x^+) = -A^+(x^+, \hat{\gamma}^+) - A^+(x^-, \hat{\gamma}^-),$$
$$2L^-(x^-) = A^-(x^+, \hat{\gamma}^-) + A^-(x^-, \hat{\gamma}^+),$$

in particular we have

$$A^+(x^-, \hat{\gamma}^-) \equiv 0 \equiv A^-(x^+, \hat{\gamma}^-),$$

or equivalently,

$$A(-, \hat{\gamma}^-) \equiv 0.$$

Since we are assuming A nondegenerate, we must have $\hat{\gamma}^- = 0$, and we have

$$
\begin{aligned}
L^+(x^+) &= -\frac{A^+}{2}(x^+, \hat{\gamma}^+) \\
L^-(x^-) &= \frac{A^-}{2}(x^-, \hat{\gamma}^+)
\end{aligned} \tag{8}
$$

We look now for respective complex structures J_1, J_2 on $\Lambda \otimes \mathbb{R}$, $\Gamma \otimes \mathbb{R}$ making L, A_1 and A_2 antiholomorphic.

The condition $J_2 A_2 = -A_2 J_2$ immediately implies that J_2 exchanges the two Eigenspaces for A_2, therefore we can write $J_2(x^+, x^-) = (Cx^-, Dx^+)$. The condition $J_2^2 = -Id$ is then equivalent to $D = -C^{-1}$, so, if we set $B_2 = -C$, we have

$$J_2(x^+, x^-) = (-B_2 x^-, B_2^{-1} x^+).$$

Proceeding analogously for J_1 we obtain

$$J_1(y^+, y^-) = (-B_1 y^-, B_1^{-1} y^+).$$

Finally $L \circ J_2 = -J_1 \circ L$ is equivalent to

$$L(-B_2 x^-, B_2^{-1} x^+) = (B_1 L^-(x), -B_1^{-1} L^+(x)),$$

equivalently,

$$
\begin{aligned}
L^+(-B_2 x^-, B_2^{-1} x^+) &= B_1 L^-(x^+, x^-), \\
L^-(-B_2 x^-, B_2^{-1} x^+) &= -B_1^{-1} L^+(x^+, x^-).
\end{aligned}
$$

If we now write $L(x^+, x^-)$ as $L_+ x^+ + L_- x^-$ we obtain

$$
\begin{aligned}
-L_+^+ B_2(x^-) + L_-^+ B_2^{-1}(x^+) &= B_1 L_+^-(x^+) + B_1 L_-^-(x^-), \\
-L_+^- B_2(x^-) + L_-^- B_2^{-1}(x^+) &= -B_1^{-1} L_+^+(x^+) - B_1^{-1} L_-^+(x^-).
\end{aligned}
$$

We rewrite the first equation as

$$
\begin{aligned}
L_+^+ B_2 + B_1 L_-^- &\equiv 0, \\
L_-^+ B_2^{-1} - B_1 L_+^- &\equiv 0.
\end{aligned} \tag{9}
$$

After rewriting the second equation as

$$L_+^- B_2 - B_1^{-1} L_-^+ \equiv 0,$$

$$L_-^- B_2^{-1} + B_1^{-1} L_+^+ \equiv 0,$$

we observe that these equations are clearly equivalent to (9).

Conditions (8) become

$$
\begin{aligned}
L_+^+(x^+) &= -\frac{A^+}{2}(x^+, \hat{\gamma}^+), \\
L_-^-(x^-) &= \frac{A^-}{2}(x^-, \hat{\gamma}^+).
\end{aligned} \tag{10}
$$

We shall now write the Riemann bilinear relations.

Recall that if $(V,U) \in \mathcal{TB}_A$, $V = \{x - iJ_2x \mid x \in \Gamma\otimes\mathbb{R}\}$, $U = \{z - iJ_1z \mid z \in \Lambda\otimes\mathbb{R}\}$ and we want

$A(x - iJ_2x, y - iJ_2y) = A(x,y) - A(J_2x, J_2y) - i(A(x, J_2y) + A(J_2x, y)) \in U$, $\forall x, y \in \Gamma\otimes\mathbb{R}$.

This means that

$$A(x, J_2y) + A(J_2x, y) = J_1A(x,y) - J_1A(J_2x, J_2y). \tag{11}$$

We shall now split (11) into U^+ and U^- components, so that using (6), (7) the first component is

$$\begin{gathered} A^+_{++}(x^+, -B_2y^-) + A^+_{--}(x^-, B_2^{-1}y^+) + \\ A^+_{++}(-B_2x^-, y^+) + A^+_{--}(B_2^{-1}x^+, y^-) = -B_1A^-_{+-}(x^+, y^-) - B_1A^-_{-+}(x^-, y^+) \\ -B_1A^-_{+-}(B_2x^-, B_2^{-1}y^+) - B_1A^-_{-+}(B_2^{-1}x^+, B_2y^-) \end{gathered} \tag{12}$$

while the second component is

$$\begin{gathered} A^-_{+-}(x^+, B_2^{-1}y^+) + A^-_{-+}(B_2^{-1}x^+, y^+) + \\ A^-_{+-}(-B_2x^-, y^-) + A^-_{-+}(x^-, -B_2y^-) = B_1^{-1}[A^+_{++}(x^+, y^+) + A^+_{--}(x^-, y^-) \\ -A^+_{++}(B_2x^-, B_2y^-) - A^+_{--}(B_2^{-1}x^+, B_2^{-1}y^+)]. \end{gathered} \tag{13}$$

From these equations we derive the following equations looking at the four possible bilinear types (x^+, y^+), (x^+, y^-), (x^-, y^+), (x^-, y^-).

$$\begin{gathered} -A^+_{++}(x^+, B_2y^-) + A^+_{--}(B_2^{-1}x^+, y^-) = \\ = -B_1A^-_{+-}(x^+, y^-) - B_1A^-_{-+}(B_2^{-1}x^+, B_2y^-) \end{gathered}$$

$$\begin{gathered} A^+_{--}(x^-, B_2^{-1}y^+) - A^+_{++}(B_2x^-, y^+) = \\ = -B_1A^-_{-+}(x^-, y^+) - B_1A^-_{+-}(B_2x^-, B_2^{-1}y^+) \end{gathered}$$

$$\begin{gathered} A^-_{+-}(x^+, B_2^{-1}y^+) + A^-_{-+}(B_2^{-1}x^+, y^+) = \\ = B_1^{-1}[A^+_{++}(x^+, y^+) - A^+_{--}(B_2^{-1}x^+, B_2^{-1}y^+)] \end{gathered}$$

$$\begin{gathered} -A^-_{+-}(B_2x^-, y^-) - A^-_{-+}(x^-, B_2y^-) = \\ = B_1^{-1}[A^+_{--}(x^-, y^-) - A^+_{++}(B_2x^-, B_2y^-)]. \end{gathered}$$

These four equations can be rewritten as tensor equations as follows (according to the standard notation for the transformation of bilinear forms)

$$-A^+_{++}B_2 +^t B_2^{-1}A^+_{--} = B_1(-A^-_{+-} -^t B_2^{-1}A^-_{-+}B_2) \tag{14}$$

$$A^+_{--}B_2^{-1} -^t B_2A^+_{++} = B_1(-A^-_{-+} -^t B_2A^-_{+-}B_2^{-1}) \tag{15}$$

$$A^-_{+-}B_2^{-1} +^t B_2^{-1}A^-_{-+} = B_1^{-1}(A^+_{++} -^t B_2^{-1}A^+_{--}B_2^{-1}) \tag{16}$$

$$-{}^tB_2A^-_{+-} - A^-_{-+}B_2 = B_1^{-1}(A^+_{--} -{}^t B_2A^+_{++}B_2) \tag{17}$$

We observe that the tensor equations above are all equal, in fact (15) yields (14) by composing with B_2 to the right and with ${}^tB_2^{-1}$ to the left. (16) yields (14) by composing with B_2 to the right. (17) yields (14) by composing with ${}^tB_2^{-1}$ to the left.

So we have shown that if we have a real structure we have only one equation for the Riemann bilinear relation:

$$A^+_{--} -{}^t B_2A^+_{++}B_2 = -B_1(A^-_{-+}B_2 -{}^t B_2\, {}^tA^-_{-+}), \tag{18}$$

where we have used $A^-_{+-} = -{}^tA^-_{-+}$ since A^- is alternating.

Now, in order to simplify the notation, we set $A^+_{--} =: A_-$, $A^+_{++} =: A_+$, $D := A^-_{-+}$, so (18) becomes

$$A_- -{}^t B_2A_+B_2 = -B_1(DB_2 -{}^t (DB_2)), \tag{19}$$

We can now easily show that in the case $m = d = 1$ the space $\mathcal{T}'\mathcal{B}^{\mathbb{R}}_A(A_1, A_2, L) \cap (\mathcal{T}_1 \times \mathcal{T}_1)$ is connected. We observe, here and in the following, that it suffices to show that $\mathcal{T}\mathcal{B}^{\mathbb{R}}_A(A_1, A_2, L) \cap (\mathcal{T}_1 \times \mathcal{T}_1)$ is connected. Note that the case $m = d = 1$, since we assume A to be nondegenerate, corresponds to the case of Kodaira surfaces. The fact that the moduli space of real Kodaira surfaces of a given topological type is connected was already proved by the second author with different methods (cf. [F04]).

Proposition 4.7. *If $m = d = 1$, the space $\mathcal{T}\mathcal{B}^{\mathbb{R}}_A(A_1, A_2, L) \cap (\mathcal{T}_1 \times \mathcal{T}_1)$ is connected.*

Proof. We observe first of all that equation (19) does not appear since for $m = 1$, $A_- = A_+ = 0$, and B_1, B_2 are scalars.

So we only have to consider conditions (9). With an appropriate choice of orientation on each vector space we may assume $B_1 > 0, B_2 > 0$, so if $L^-_- \neq 0$, by (9) we obtain $B_1 = -\frac{B_2L^+_+}{L^-_-}$ and

$$L^+_- + \frac{B_2^2L^+_+L^-_+}{L^-_-} = 0,$$

or equivalently

$$L^+_-L^-_- + B_2^2L^+_+L^-_+ = 0,$$

which is solvable if either $L^+_-L^-_- = L^+_+L^-_+ = 0$, or both are non zero and have opposite sign and in this case we only have one positive solution.

If $L^-_- = 0$, then also $L^+_+ = 0$ and we have $B_1B_2 = \frac{L^+_-}{L^-_+}$. Since we are assuming $B_1 > 0$, $B_2 > 0$, we must have $\frac{L^+_-}{L^-_+} > 0$ and the set is clearly connected. □

Remark 4.8. *Observe that if $L = 0$ and $d = 1$, we set $B_2 =: B$, $B_1 =: b$, a scalar, and we only have the equation*

$$A_- -{}^t BA_+B = -b(DB -{}^t (DB)), \tag{20}$$

If we also assume $m = 2$, we can write $A_- = a_- \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$,

$A_+ = a_+ \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$, *and we obtain only one scalar equation*

$$a_- - det(B)a_+ = -b(d_{11}b_{12} + d_{12}b_{22} - d_{21}b_{11} - d_{22}b_{21})$$

and we must combine it with the condition

$$det(B) = b_{11}b_{22} - b_{12}b_{21} > 0.$$

Theorem 4.9. *For $d = 1$, $m = 2$, the space $\mathcal{X} = \mathcal{TB}_A^{\mathbb{R}}(A_1, A_2, L) \cap (\mathcal{T}_2 \times \mathcal{T}_1)$ is connected.*

Proof. By remark (4.8), if $L = 0$ we have the two conditions:

(21) $$a_- - det(B)a_+ = -b(d_{11}b_{12} + d_{12}b_{22} - d_{21}b_{11} - d_{22}b_{21})$$

(22) $$det(B) = b_{11}b_{22} - b_{12}b_{21} > 0,$$

and we assume $b > 0$.

Assume that the matrix $D = 0$: then we have a product of the half-line $\{b > 0\}$ with the quadric $\{detB = a\}$, where a is a positive constant. In this case we are done since $\{detB = a\}$ is a central quadric in $\mathbb{R}^4$ with quadratic part of signature $(+, +, -, -,)$, and is therefore connected.

Assume then that $D \neq 0$, then the rigth hand side is equal to bb', where we define b' as the linear form $b' := d_{11}b_{12} + d_{12}b_{22} - d_{21}b_{11} - d_{22}b_{21}$.

We want to change coordinates in the $\mathbb{R}^4$ of the matrices B, completing the linear form b' to a basis (b', x, y, z) of the space of linear forms. The projective quadric $\{detB = 0\}$ and the hyperplane $\{b' = 0\}$ determine an affine quadric in $\mathbb{R}^3$ with the given signature, therefore there are only two cases, according to the property whether the hyperplane is transversal or tangent to the quadric.

In the first case, after dehomogenizing (i.e., setting $b' = 1$) we get a one sheeted (hyperbolic) hyperboloid:

$$\pm detB = {b'}^2 - z^2 + xy,$$

in the second case we get a hyperbolic paraboloid

$$detB = b'z + xy.$$

Let us observe that the space $\mathcal{X}$ maps to the open set Ω in $\mathbb{R}^4$ defined by the inequality $detB > 0$, and that if we define $\Omega' := \Omega \cap \{b' \neq 0\}$, then Ω' is homeomorphic to its inverse image in $\mathcal{X}$.

It is immediate to conclude, in view of the above normal forms, that the open sets $\Omega^+ := \Omega \cap \{b' > 0\}$, resp. $\Omega^- := \Omega \cap \{b' < 0\}$ are connected.

We also see by the way (this remark is not indispensable) that the inverse image $\mathcal{X}^0$ of $\{b' = 0\}$ has at most two connected components, since it is the

product of the half-line $\{b > 0\}$ with a central quadric $\{detB = a > 0\} \cap \{b' = 0\}$ in $\mathbb{R}^3$ with possible signatures $(+,+,-)$,$(+,-,-)$,$(+,-)$.

In order to prove that $\mathcal{X}$ is connected, let us observe that the space $\mathcal{X}$ we are considering is the intersection of a real quadric Q in $\mathbb{R}^5$ with an open set. Moreover, the quadric Q is centred in the origin and the associated quadratic form has negativity index, respectively positivity index, at least two. Therefore Q is everywhere of pure dimension 4, thus the closed set $\mathcal{X}^0$ is in the closure of the complement $\mathcal{X} \setminus \mathcal{X}^0$, and it suffices to show that there is a point p of $\mathcal{X}^0$ and a neighbourhood of p meeting both $\mathcal{X}^+$ and $\mathcal{X}^-$.

But if this were not so, in the points of $\mathcal{X}^0$ the linear form b' restricted to $\mathcal{X}$ would vanish with its derivatives everywhere, which implies that $\mathcal{X}^0$ would be a linear subspace counted with multiplicity two, which is not the case.

We have thus shown that if $L = 0$, $\mathcal{X}$ is connected.

Assume now that $L \neq 0$. In this case we also have to consider equations (9):

$$L_+^+ B = -bL_-^-,$$
$$L_+^- B = b^{-1} L_-^+.$$

If L_+^+, L_+^- are linearly independent we can uniquely determine B

$$B = \begin{pmatrix} L_+^+ \\ L_+^- \end{pmatrix}^{-1} \begin{pmatrix} -bL_-^- \\ b^{-1}L_-^+ \end{pmatrix},$$

where $\begin{pmatrix} L_+^+ \\ L_+^- \end{pmatrix}$ denotes the (2×2) matrix whose first row is L_+^+ and whose second row is L_+^- and $\begin{pmatrix} -bL_-^- \\ b^{-1}L_-^+ \end{pmatrix}$ is the (2×2) matrix whose first row is $-bL_-^-$ and whose second row is $b^{-1}L_-^+$.

Then $det(B) = -\dfrac{det(\begin{pmatrix} L_-^- \\ L_-^+ \end{pmatrix})}{det(\begin{pmatrix} L_+^+ \\ L_+^- \end{pmatrix})} =: \alpha.$

So we must have $\alpha > 0$, and equation (21) is of the form

$$a_- - \alpha a_+ = -b(c_1 b + c_2 b^{-1}),$$

where c_1, c_2 are functions depending only on D and L.

So we have to solve an equation of the form

$$c_1 b^2 + c = 0,$$

and we have at most one positive solution, thus $\mathcal{X}$ is connected.

Suppose now $L_+^- = 0$. Then if there is a solution also $L_-^+ = 0$ and we may assume without loss of generality that $L_+^+ = (1, 0)$ (since $L \neq 0$).

If we then set $B = (b_{ij})$ our equations (9) reduce to

$$(b_{11}, b_{12}) = -bL_-^-,$$

and equation (21) becomes

$$a_- - a_+ det(B) = -b(c_0 b_{21} + c_2 b_{22} + bc_1),$$

where the c_j's only depend on D and L.

Moreover, we can write $det(B) = b\rho$, where $\rho = l_0 b_{21} + l_2 b_{22}$ and the l_j's depend only on L. So we have an equation in b, b_{21}, b_{22}

$$a_- - a_+ b\rho = -b(c_0 b_{21} + c_2 b_{22} + bc_1),$$

which is an equation of the form

$$b^2 c_1 + b(a_0 b_{21} + a_2 b_{22}) + a_- = 0,$$

where the a_j's depend only on L and D.

The condition $det(B) > 0$ becomes $\rho > 0$, which is a linear inequality in b_{21}, b_{22}. If we fix $b \in \mathbb{R}_+$, our solution is the intersection of the line in $\mathbb{R}^2$ given by

$$a_0 b_{21} + a_2 b_{22} + \frac{a_- + b^2 c_1}{b} = 0$$

with the half plane $\rho > 0$.

To simplify things, we introduce a linear form τ in the $\mathbb{R}^2$ with coordinates b_{21}, b_{22}, namely

$$\tau := a_0 b_{21} + a_2 b_{22}.$$

There are three cases:

1) the linear forms ρ, τ are independent

2) $\tau = c\rho$ for a constant $c \neq 0$.

3) $\tau = 0$.

In case 1) we have $\tau = -\frac{a_- + b^2 c_1}{b}$, $\rho > 0, b > 0$, and $\mathcal{X}$ is homeomorphic to a quadrant in $\mathbb{R}^2$.

In case 2) we have that $\mathcal{X}$ is a product of $\mathbb{R}$ with the set $\{(\rho, b) | \rho > 0, b > 0, \rho = -\frac{a_- + b^2 c_1}{bc}\} \subset \mathbb{R}^2$, which is diffeomorphic to the interval $\{(b) | b > 0, 0 > c(a_- + b^2 c_1)\} \subset \mathbb{R}$.

In case 3) $\mathcal{X}$ is a product of $\mathbb{R}$ with the set

$$\{(\rho, b) | \rho > 0, b > 0, 0 = a_- + b^2 c_1\} \subset \mathbb{R}^2.$$

which is a half-line in $\mathbb{R}^2$,

Therefore $\mathcal{X}$ is connected in all three cases.

Assume now $L_+^- \neq 0$, $L_+^+ = \beta L_+^-$, $\beta \neq 0$. Then

$$L_+^+ B = -b L_-^-,$$

$$L_+^- B = b^{-1} L_-^+,$$

yield an equation of the form

$$b^2 L_-^- = -\beta L_-^+,$$

and if we solve for b we find at most one positive solution $\hat{b}$.

Without loss of generality we can assume $L_+^+ = (1, 0)$ and the equation $L_+^+ B = -\hat{b} L_-^-$ allows us to determine the first row of B.

Hence both $det(B)$ and equation (21) are polynomials of degree one in b_{21}, b_{22}, therefore the intersection of $\{det(B) > 0\}$ with the set where the equation (21) is satified is connected (a half-line) or empty.

Finally, if $L_+^+ = 0$, then also $L_-^- = 0$ and we may assume w.l.o.g. $L_+^- = (1, 0)$. Therefore we have

$$(b_{11}, b_{12}) = b^{-1} L_-^+ =: b^{-1}(l_1, l_2),$$

and since then $a_- - det(B)a_+ = a_- - a_+ b^{-1}(l_1 b_{22} - l_2 b_{21})$ our equation becomes

$$a_- - a_+ b^{-1}(l_1 b_{22} - l_2 b_{21}) = -b(b^{-1}(d_{11} l_2 - d_{21} l_1) + d_{12} b_{22} - d_{22} b_{21}),$$

and the condition $det(B) > 0$ can be written as $l_1 b_{22} - l_2 b_{21} > 0$.

So we get

$$b_{22}(d_{12} b^2 - a_+ l_1) + b_{21}(-b^2 d_{22} + a_+ l_2) = -b(a_- - d_{21} l_1 + d_{11} l_2),$$
$$l_1 b_{22} - l_2 b_{21} > 0.$$

We define functions

$$x(b_{22}, b_{21}) := l_1 b_{22} - l_2 b_{21}$$
$$y(b, b_{22}, b_{21}) := b_{22}(d_{12} b^2 - a_+ l_1) + b_{21}(-b^2 d_{22} + a_+ l_2).$$

These are linear functions of (b_{22}, b_{21}) with determinant $b^2(d_{12} l_2 - d_{22} l_1)$.

Case 1) : $(d_{12} l_2 - d_{22} l_1) \neq 0$.

Then $(b, b_{22}, b_{21}) \to (b, x(b_{22}, b_{21}), y(b, b_{22}, b_{21}))$ is a self-diffeomorphism of $\mathbb{R}_+ \times \mathbb{R}^2$, which transforms $\mathcal{X}$ into the set $\mathcal{X}' \subset \mathbb{R}^3$, $\mathcal{X}' = \{(b, x, y) | b > 0, x > 0, y = -b(a_- - d_{21} l_1 + d_{11} l_2)\}$, which is diffeomorphic to the quadrant $\{(b, x) | b > 0, x > 0\}$, which is clearly connected.

Case 2) : $(d_{12} l_2 - d_{22} l_1) = 0$.

In this case

$$y(b, b_{22}, b_{21}) := b_{22}(d_{12} b^2 - a_+ l_1) + b_{21}(-b^2 d_{22} + a_+ l_2) = (cb^2 - a_+)\ x(b_{22}, b_{21}).$$

There is a linear self-diffeomorphism of $\mathbb{R}^3$ such that

$$(b, b_{22}, b_{21}) \to (b, x(b_{22}, b_{21}), z(b_{22}, b_{21}))).$$

It carries $\mathcal{X}$ into the set

$\{(b, x, z) | b > 0, x > 0, (cb^2 - a_+)x = -b(a_- - d_{21} l_1 + d_{11} l_2)\}$, which is the product of $\mathbb{R}$ with the subset of $\mathbb{R}^2$

$$\mathcal{Y} := \{(b, x) | b > 0, x > 0, (cb^2 - a_+)x = -b(a_- - d_{21} l_1 + d_{11} l_2)\}.$$

Case 2.1) : $(a_- - d_{21} l_1 + d_{11} l_2) = 0$.

Then we get $(cb^2 - a_+) = 0$, which has at most one positive solution, and $\mathcal{Y}$ is diffeomorphic to the half-line $\{x > 0\}$.

Case 2.2) : $(a_- - d_{21} l_1 + d_{11} l_2) \neq 0$, $c = 0$.

In this subcase either $\mathcal{Y}$ is the empty set, or $\mathcal{Y}$ is diffeomorphic to $\{(b \in \mathbb{R}) | b > 0\}$.

Case 2.3) : $(a_- - d_{21} l_1 + d_{11} l_2) \neq 0$, $c \neq 0$.

In this subcase we get that $\mathcal{Y}$ is diffeomorphic to the set $\{b | b > 0,\ cb^2 < a_+\}$ or to the set $\{b | b > 0,\ cb^2 > a_+\}$.

We are done, since in both cases either $\mathcal{Y}$ is empty or it is diffeomorphic to an interval.

□

From the above theorem follows now easily

Theorem C *Assume again $d = 1, m = 2$, that A is nondegenerate and fix the orbifold fundamental group Π_σ. Then the real structures for torus bundles in the Appell Humbert family are parametrized by a connected family.*

Proof. It suffices to combine the connectedness result that we have just proven in Theorem 4.9 with theorem 4.3, with Remark 4.4, and with the fact that $\mathcal{T}'\mathcal{B}_A^{\mathbb{R}}(A_1, A_2, L)$ is connected if and only if $\mathcal{T}\mathcal{B}_A^{\mathbb{R}}(A_1, A_2, L)$ is connected.

□

As an immediate corollary we have

Theorem D *Same assumptions as in theorem B: $d = 1, m = 2$, and i), ii), iii) are satisfied.*

Then simplicity holds for real torus bundles in the Appell Humbert family.

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